

## A REMARK ON THE LEFSCHETZ FIXED POINT FORMULA FOR DIFFERENTIABLE MAPS

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If 0 is an isolated fixed point for the continuous map  $f: U \rightarrow R^m$ , where  $U$  is an open subset of  $R^m$ , then the index of  $f$  at 0,  $\sigma_f(0)$ , is the local degree of the mapping  $\text{Id}-f$  restricted to an appropriately small open set about 0. If 0 is an isolated fixed point of  $f^n$ , then  $\sigma_{f^n}(0)$  is defined for all  $n > 0$ , where  $f^n$  means  $f$  composed with itself  $n$  times restricted to a small neighborhood of 0. We will use a little elementary calculus to show:

**PROPOSITION.** *Suppose that  $f: U \rightarrow R^m$  is  $C^1$  and that 0 is an isolated fixed point of  $f^n$  for all  $n$ . Then  $\sigma_{f^n}(0)$  is bounded as a function of  $n$ .*

The proposition is not true for continuous functions as the mapping of the complex plane  $f(z) = 2z^2/\|z\|$  shows. In fact, for this  $f$ ,  $\sigma_{f^n}(0) = 2^n$ . Our interest in the proposition arose from the Lefschetz fixed point formula as applied to a smooth endomorphism  $f$  of a compact differentiable manifold  $M$ . The Lefschetz formula says that the Lefschetz numbers

$$L(f^n) \equiv \sum (-1)^i \text{trace } f_{*i}^n: H_i M \rightarrow H_i M$$

can be computed locally by these fixed point indices,

$$L(f^n) = \sum_{P \in \text{Fix } f^n} \sigma_{f^n}(P),$$

provided that the fixed points of  $f^n$  are isolated.

**COROLLARY.** *If  $f: M \rightarrow M$  is  $C^1$ , and the Lefschetz numbers  $L(f^n)$  are not bounded then the set of periodic points of  $f$  is infinite.*

In particular, any  $C^1$  degree two map of the two sphere,  $S^2$ , has an infinite number of periodic points and hence an infinite non-wandering set [see 1].<sup>†</sup> The corollary suggests the possibility of getting sharper estimates on the asymptotic growth rate of  $N_n(f)$ , the number of fixed points of  $f^n$ .

*Problem.* If  $f: M \rightarrow M$  is smooth, is

$$\limsup \frac{1}{n} \log |L(f^n)| \leq \limsup \frac{1}{n} \log N_n(f)?$$

<sup>†</sup> Note that the one-point compactification of  $f(z) = 2z^2/\|z\|$  is a continuous degree two map of  $S^2$  with only two periodic points.

As remarked in [1] this inequality is rather obviously true for the set of  $C^r$  endomorphisms  $f$  of  $M$  which have the property that all periodic points of  $f$  are transversal. Then, of course,  $|L(f^n)| \leq N_n(f)$ .

We now proceed with the proof of the proposition. In all that follows below  $f$  is  $C^1$  and 0 is an isolated fixed point of  $f^n$  for all  $n$ . The idea is to try to approximate  $I - f^n$  by  $(I + f + f^2 + \cdots + f^{n-1})(I - f)$  so that if  $I + f + f^2 + \cdots + f^{n-1}$  is a local diffeomorphism then  $\text{degree}(I - f^n) = \pm \text{degree}(I - f)$ . To make this precise and to do the estimates we work with the derivatives of  $f^n$  at 0 which we denote by  $Df^n$ .

LEMMA 1. *If  $\sum_{j=0}^{n-1} Df^j$  is non-singular then  $\sigma_f(0) = \pm \sigma_{f^n}(0)$ .*

Before we prove Lemma 1 we will show how it proves the proposition.  $\sum_{j=0}^{n-1} Df^j$  is singular precisely when  $n = mk$ ,  $k > 1$ , and  $Df$  has a primitive  $k$ th root of unity as an eigenvalue. For each integer  $n$ , let  $\lambda$  be the least common multiple of these orders  $k$ . Then we may apply the proposition to see that  $\sigma_{f^n}(0) = \pm \sigma_{f^\lambda}(0)$ . (If  $(k_1, k_2, \dots)$  are the orders of roots of unity in the spectrum of  $Df$ , then  $(k_1/\text{g.c.d.}(k_1, \lambda), \dots)$  are the orders for  $Df^\lambda$ . But now  $n/\lambda$  is not a multiple of any of these orders greater than 1.)

Since we only need finitely l.c.m.'s  $\lambda$  to take care of all the integers  $n$ , this argument proves the proposition.

A standard fact that we shall use in proving Lemma 1 is:

LEMMA 2. *If  $h, k: U \rightarrow R^n$  are continuous, have 0 as an isolated 0 and  $\|h(x) - k(x)\| < \|h(x)\|$  then  $\text{degree}(h) = \text{degree}(k)$ .*

*Proof of Lemma 1.* Let  $f = Df + \theta_1$  and  $f^n = Df^n + \theta_n$ .

Then  $I - f^n = I - Df^n - \theta_n$

$$\begin{aligned} &= (I + Df + \cdots + Df^{n-1})(I - Df) - \theta_n \\ &= (I + Df + \cdots + Df^{n-1})(I - f) + (I + Df + \cdots + Df^{n-1})\theta_1 - \theta_n. \end{aligned}$$

We will show by induction that given  $(n, \epsilon)$  there is a neighborhood  $U_{n, \epsilon}$  of 0 such that

$$\left\| \left( \sum_{j=0}^{n-1} Df^j \theta_1 - \theta_n \right) (x) \right\| < \epsilon \| (I - f)(x) \| \text{ for all } x \in U_{n, \epsilon}.$$

So that if  $\sum_{j=0}^{n-1} Df^j$  is non-singular then by Lemma 2,

$$\text{degree}(I - f^n) = \text{degree} \left( \sum_{j=0}^{n-1} Df^j \right) (I - f) = \pm \text{degree}(I - f).$$

To estimate  $\sum_{j=0}^{n-1} Df^j \theta_1 - \theta_n$  first observe that  $\theta_n = \sum_{j=0}^{n-1} Df^{n-j-1} \theta_1 f^j$  as can easily be seen by induction. So

$$\begin{aligned} \sum_{j=0}^{n-1} Df^j \theta_1 - \theta_n &= \sum_{j=0}^{n-1} Df^{n-1-j} \theta_1 - \sum_{j=0}^{n-1} Df^{n-j-1} \theta_1 f^j \\ &= \sum_{j=1}^{n-1} Df^{n-1-j} (\theta_1 - \theta_1 f^j). \end{aligned}$$

By the mean value theorem

$$\left\| \left( \sum_{j=0}^{n-1} Df^j \theta_1 - \theta_n \right) (x) \right\| \leq \sum_{j=1}^{n-1} \| Df^{n-1-j} \| \| D\theta_1 \|_{U_{n, \epsilon}} \| (I - f^j)(x) \|$$

where  $\|D\theta_1\|_{U_{n,\varepsilon}} = \sup_{x \in U_{n,\varepsilon}} \|D_x \theta_1\|$ . Since  $D_0 \theta_1 = 0$  it clearly suffices to prove inductively

that given  $j < n$  there is a neighborhood  $V_j$  of 0 and a  $0 \leq k_j < \infty$  such that

$$\|(I - f^j)(x)\| \leq k_j \|(I - f)(x)\| \quad \text{for all } x \in V_j.$$

Since

$$I - f^j = \left( \sum_{i=0}^{j-1} Df^i \right) (I - f) + \sum_{i=0}^{j-1} Df^i \theta_1 - \theta_j,$$

we can inductively choose  $U_{j,\varepsilon}$  so that

$$\|(I - f^j)(x)\| \leq \sum_{i=0}^{j-1} \|Df^i\| \|(I - f)(x)\| + \varepsilon \|(I - f)(x)\|,$$

and we are done.

#### REFERENCE

1. M. SHUB: Dynamical systems, filtrations and entropy, *Bull. Am. math. Soc.* **80** (1974), 27-41.

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